Lec 3

I A set of vectors

$$v_1, v_2, \ldots, v_m$$

are linearly independent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$$
$$\Updownarrow$$
$$\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$$

A set of vectors

$$v_1, v_2, \ldots, v_m$$

are linearly dependent if $\exists\ \alpha_1, \ldots, \alpha_m$ not all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$$

Linear combination:

Let $v_1, \ldots, v_m$ be a set of vectors
$\alpha_1, \ldots, \alpha_m$ be a set of scalars.
then $\alpha_1 v_1 + \cdots + \alpha_m v_m$ is called a linear combination of the vectors $v_1, \ldots, v_m$.

Span:

Let $v_1, \ldots, v_m$ be a set of vectors.

Let $\mathcal{S}$ be the set of all linear combinations of $v_1, \ldots, v_m$.

$$\mathcal{S} = \{\alpha_1 v_1 + \cdots + \alpha_m v_m : \alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}\}$$

$\mathcal{S}$ is called the span of vectors $v_1, \ldots, v_m$. It is denoted as

$$\mathcal{S} = [v_1, v_2, \ldots, v_m].$$

Fact: Let $v_1, v_2, \ldots, v_m$ be a set of vectors in V. Then span of $v_1, \ldots, v_m$

$$[v_1, \ldots, v_m]$$

is a vector subspace of V.

Ex: 1

Let
$$V_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad V_2 = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$$

be two vectors in $\mathbb{R}^2$

$$[V_1 \ \ V_2] = \{\alpha_1 V_1 + \alpha_2 V_2 : \alpha_1, \alpha_2 \in \mathbb{R}\}$$
$$= \{(\alpha_1 - 2\alpha_2, 3\alpha_1 - 6\alpha_2) : \alpha_1, \alpha_2 \in \mathbb{R}\}$$

$\boxed{\text{Define } \beta = \alpha_1 - 2\alpha_2}$

$$= \{(\beta, 3\beta) : \beta \in \mathbb{R}\}$$
$$= [V_1]$$

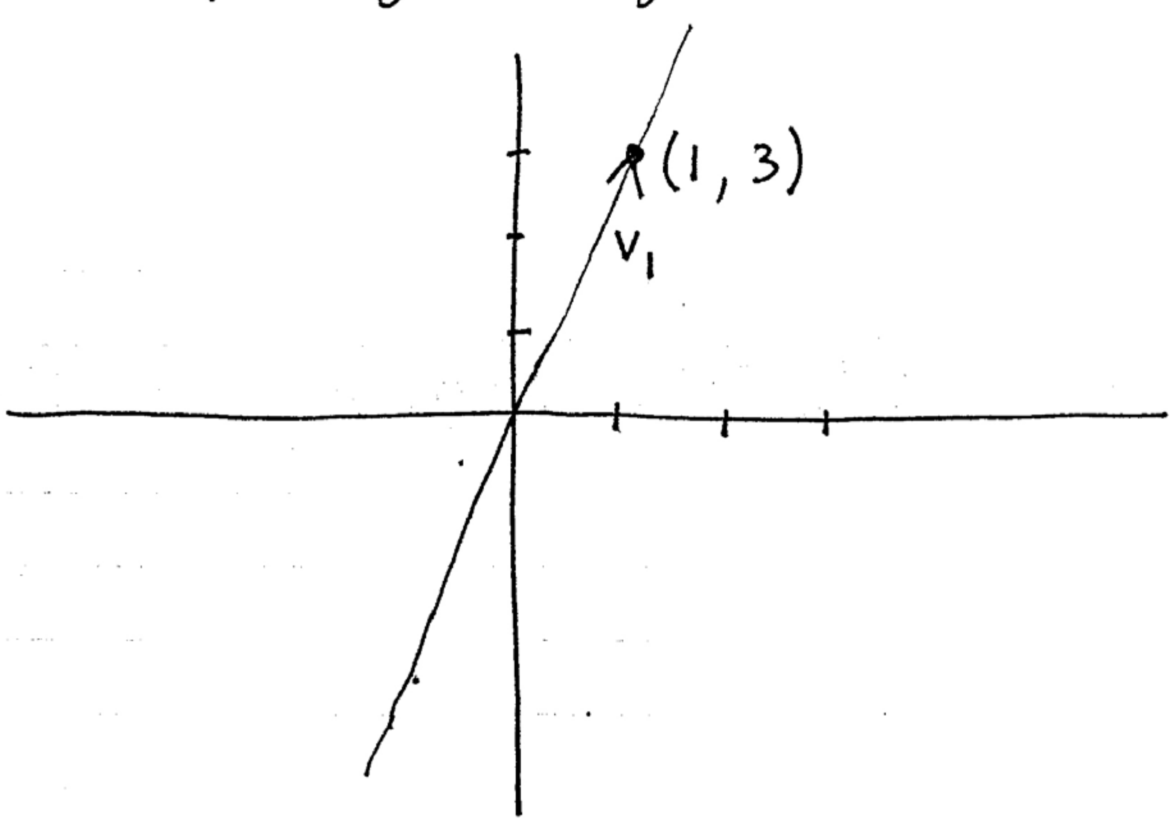
We showed that span of V_1 and V_2 is same as span of V_1. Actually it is also same as span of V_2.

Thus, in a certain sense, we dont need two vectors to describe

$$\left[\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -6 \end{pmatrix} \right]$$

Only one vector is enough.

Geometrically, $[v_1]$ is a homogeneous line passing through v_1



Ex: 2

Check if the vector

$u = (3 \quad 7 \quad -9)$

Is contained in the span of v_1 and v_2 and v_3

where

$v_1 = (1 \quad 5 \quad 8)$

$v_2 = (3 \quad 15 \quad 24)$

$v_3 = (2 \quad 6 \quad -1)$

Sol^n: If $u \in [v_1 \quad v_2 \quad v_3]$ then

$\exists\, \alpha_1, \alpha_2, \alpha_3$:

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$\Rightarrow (\alpha_1 \quad 5\alpha_1 \quad 8\alpha_1) + (3\alpha_2 \quad 15\alpha_2 \quad 24\alpha_2) + (2\alpha_3 \quad 6\alpha_3 \quad -\alpha_3)$$

$$= (3 \quad 7 \quad -9)$$

$$\Rightarrow \alpha_1 + 3\alpha_2 + 2\alpha_3 = 3 \qquad (A)$$

$$5\alpha_1 + 15\alpha_2 + 6\alpha_3 = 7 \qquad (B)$$

$$8\alpha_1 + 24\alpha_2 - \alpha_3 = -9 \qquad (C)$$

(B) − 5(A) :

$$-4\alpha_3 = -8 \quad \Rightarrow \boxed{\alpha_3 = 2} \quad (E)$$

(C) − 8(A) :

$$-17\alpha_3 = -33 \quad \Rightarrow \boxed{\alpha_3 = \frac{33}{17}} \quad (F)$$

$\therefore$ (E) and (F) given two different values of α_3. Hence (A), (B), (C) cannot be solved.

Thus we conclude that u is not contained in the span of v_1, v_2, v_3.

Ex: 3

Check if the vector

$u = (3 \quad 7 \quad -10)$

is contained in $[v_1 \quad v_2 \quad v_3]$ where

v_1, v_2, v_3 are as given in Ex 2 page 3.5.

Sol^n:

Proceeding as in example 2, we need to solve

$$\alpha_1 + 3\alpha_2 + 2\alpha_3 = 3 \qquad (A)$$

$$5\alpha_1 + 15\alpha_2 + 6\alpha_3 = 7 \qquad (B)$$

$$8\alpha_1 + 24\alpha_2 - \alpha_3 = -10 \qquad (C)$$

Substituting $\alpha_1 + 3\alpha_2 = \beta$ we obtain

$$\left.\begin{array}{l} \beta + 2\alpha_3 = 3 \\ 5\beta + 6\alpha_3 = 7 \\ 8\beta \cancel{+} - \alpha_3 = -10 \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} \alpha_3 = 2 \\ \beta = -1 \end{array}}$$

Thus u is in the span of v_1, v_2, v_3

Further facts about examples 2 and 3.

1. The span of v_1, v_2, v_3 is a homogeneous plane in $\mathbb{R}^3$ described by

$$\boxed{53x - 17y + 4z = 0}$$

The vector u in Ex 2 is not contained in this plane. The vector u in Ex 3 is contained in the plane.

Basis of a vector space :-

Let $v_1, \ldots, v_m$ be a set of vectors in a vector space V. The vectors are said to form a basis if

(1) The vectors $v_1, \ldots, v_m$ are l. i.

(2) The span $[v_1, \ldots, v_m]$ is the vector space V.

Ex 4:

In $\mathbb{R}^2$, the vectors

$$v_1 = (1 \quad 0) \ , v_2 = (0 \quad 1)$$

form a basis. Note that v_1 & v_2 are l. i. (because the vectors are not both contained in a homogeneous line). Also, every vector $(x \quad y)$ in $\mathbb{R}^2$ can be written as

$$(x \quad y) = x\, v_1 + y\, v_2 .$$

It follows that

$$\mathbb{R}^2 = [v_1 \;\; v_2]$$

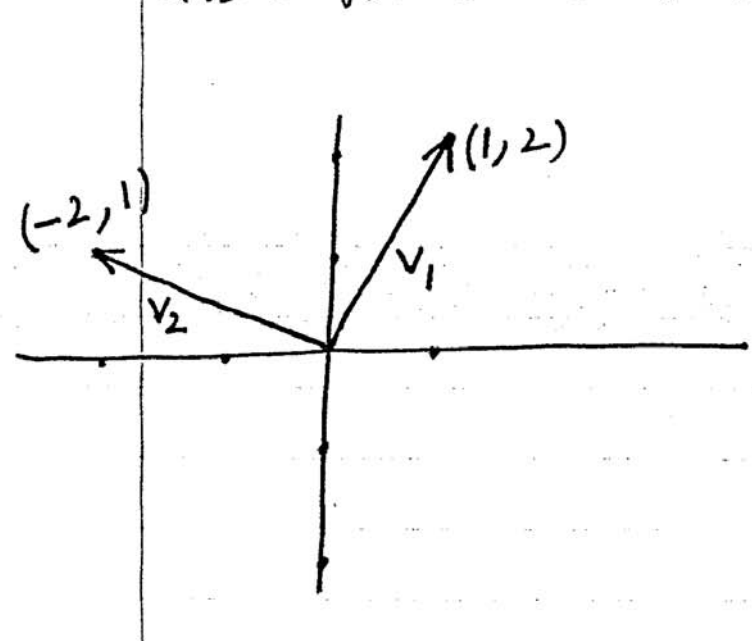
Thus v_1, v_2 is a basis of $\mathbb{R}^2$.

Remark:

~~The~~ Set of basis vectors is not unique. In fact the vectors.

$$v_1 = (1 \quad 2) \;,\; v_2 = (-2 \quad 1)$$

also forms a basis of $\mathbb{R}^2$.



The vectors v_1, v_2 are independent because they are not contained in a single homogeneous line.

with a little bit of algebra we can see that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{1}{5}x + \frac{2}{5}y\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \left(\frac{1}{5}y - \frac{2}{5}x\right)\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = r_1 v_1 + r_2 v_2$$

where $r_1 = \frac{1}{5}x + \frac{2}{5}y$

$r_2 = \frac{1}{5}y - \frac{2}{5}x$

r_1 and r_2 are called the two co-ordinates of $(x \;\; y)$ with respect to the basis vectors $(1 \;\; 2), (-2 \;\; 1)$.

Remark:

The vectors v_1 and v_2 in $\mathbb{R}^2$ in Ex 1 do not form a basis of $\mathbb{R}^2$.

This is because v_1 & v_2 are not linearly independent. This follows from the fact that

$$v_2 = -2v_1$$

or

$$2v_1 + v_2 = 0$$

There is a non trivial l.c. giving rise to the zero vector.

v_1 and v_2 are not l.i. because $\exists$ a homogeneous line

$$-3x + y = 0$$

that contains both the vectors v_1 & v_2.

Remark:

The set of vectors v_1, v_2, v_3 in $\mathbb{R}^3$ in Ex 2, do not form a basis of $\mathbb{R}^3$.

This is because v_1, v_2, v_3 are not l.i. In fact

$$v_2 - 3v_1 = 0$$

or

$$3v_1 + v_2 = 0$$

Also because the span $[v_1, v_2, v_3]$ is not the whole of $\mathbb{R}^3$ because we have a vector u in Ex 2 which is not contained in the span.

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Although the set of basis vectors in $\mathbb{R}^n$ is not unique, the following is true:

Fact I "Every set of basis vectors in $\mathbb{R}^n$ has exactly n vectors".

Fact II

"Every set of n l. i. vectors in $\mathbb{R}^n$ would form a basis".

Let us look at facts I and II for $\mathbb{R}^3$.

The general situation is analogous.

"Every set of basis vectors in $\mathbb{R}^3$ has exactly 3 vectors"

There are two parts to the above statement

I. <u>If there are four vectors v_1, v_2, v_3, v_4 in $\mathbb{R}^3$. They cannot form a basis.</u>

This is because the set $\{v_1, v_2, v_3, v_4\}$ cannot be l.i. To see this let us write

$$v_i = (a_i \quad b_i \quad c_i) ; \quad i = 1, 2, 3, 4$$

If

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$$

it follows that

$$\begin{aligned} \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \alpha_4 a_4 &= 0 \\ \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 + \alpha_4 b_4 &= 0 \\ \alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3 + \alpha_4 c_4 &= 0 \end{aligned} \qquad (*)$$

In (*) we have three equations in 4 variables $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.
We claim that there would always exist a nontrivial solution of (*) nomatter what the vectors v_1, v_2, v_3, v_4 are. Hence v_1, v_2, v_3, v_4 cannot be l. i.

II. If there are two vectors v_1, v_2 in $\mathbb{R}^3$, they cannot span $\mathbb{R}^3$.
Let us consider two vectors v_1, v_2 in $\mathbb{R}^3$. The span $[v_1 \ v_2]$ is a homogeneous plane in $\mathbb{R}^3$ described as follows:

$$v_1 = (a_1 \quad b_1 \quad c_1)$$
$$v_2 = (a_2 \quad b_2 \quad c_2)$$

(3.17)

The equation of the homogeneous plane is given by

$$\boxed{(b_1 c_2 - b_2 c_1)x + (c_1 a_2 - a_1 c_2)y + (a_1 b_2 - a_2 b_1)z = 0}$$

Of course to find the above plane I have used a bit of the cross product technology. But the point is that if $[V_1, V_2]$ is a homogeneous plane in $\mathbb{R}^3$, it definitely cannot be the whole of $\mathbb{R}^3$.

Thus it follows that every set of basis vectors of $\mathbb{R}^3$ has to have precisely 3 vectors.

Let us now look at Fact II.

"Every set of 3 l.i. vectors in $\mathbb{R}^3$ would form a basis"

Let v_1, v_2, v_3 be such a set of l.i. vectors. All what we need to show is that

$$[v_1 \quad v_2 \quad v_3] = \mathbb{R}^3.$$

Thus for any vector u in $\mathbb{R}^3$ we need to find $\alpha_1, \alpha_2, \alpha_3$: $u = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$.

Define as before

$$v_i = (a_i \quad b_i \quad c_i), \quad i = 1, 2, 3$$

and

$$u = (a \quad b \quad c)$$

We need to solve

$$\left.\begin{aligned} \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 &= a \\ \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 &= b \\ \alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3 &= c \end{aligned}\right\} \quad (\bigstar)$$

$\because$ $(\bigstar)$ represent 3 equations in 3 unknowns they can always be solved.

Hence

$$[v_1 \quad v_2 \quad v_3] = \mathbb{R}^3.$$

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In fact $\because$ v_1, v_2, v_3 are linearly independent, it follows that $(\bigstar)$ can be solved uniquely, i.e. $\alpha_1, \alpha_2, \alpha_3$ is unique. This is because if $\alpha_1', \alpha_2', \alpha_3'$ is another solution of $(\bigstar)$ we have

$$\left.\begin{aligned} \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 &= u \\ \alpha_1' v_1 + \alpha_2' v_2 + \alpha_3' v_3 &= u \end{aligned}\right\} \Rightarrow \begin{aligned} &(\alpha_1 - \alpha_1') v_1 + \\ &(\alpha_2 - \alpha_2') v_2 + \\ &(\alpha_3 - \alpha_3') v_3 = 0 \end{aligned}$$

Because v_1, v_2, v_3 are l.i., it follows that

$$\alpha_1 - \alpha_1' = 0, \quad \alpha_2 - \alpha_2' = 0, \quad \alpha_3 - \alpha_3' = 0.$$

Thus the solution of (☆) is unique

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This is good news because if v_1, v_2, v_3 is a basis of $\mathbb{R}^3$ then the co-ordinates $\alpha_1, \alpha_2, \alpha_3$ of u with respect to the basis v_1, v_2, v_3 is unique.

Dimension :

$B \triangleq$

Let $\{v_1, \dots, v_m\}$ be a basis of a vector space V. Although B is not an unique set but the integer m "which is the number of elements in any basis of V" is unique.

This m is called the dimension of V.

Dimension of $\mathbb{R}^2$ is 2

$\mathbb{R}^3$ is 3

and for that matter

$\mathbb{R}^n$ is n.

Ex 5:

Let V be the set of all vectors $(x\ \ y\ \ z\ \ w)$ in $\mathbb{R}^4$ such that $x+y+z+w=0$. Calculate the dimension of V by obtaining one set of basis vectors.

Sol^n: The set of all vectors in V are of the form

$$u = (x \quad y \quad z \quad -x-y-z)$$

where we can write

$$u = x(1 \quad 0 \quad 0 \quad -1) +$$
$$y(0 \quad 1 \quad 0 \quad -1) +$$
$$z(0 \quad 0 \quad 1 \quad -1)$$

Clearly $V = [v_1 \quad v_2 \quad v_3]$

where

$$v_1 = (1 \quad 0 \quad 0 \quad -1)$$
$$v_2 = (0 \quad 1 \quad 0 \quad -1)$$
$$v_3 = (0 \quad 0 \quad 1 \quad -1).$$

We claim that v_1, v_2, v_3 are also l.i.. This is because

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\Rightarrow (\alpha_1 \quad 0 \quad 0 \quad -\alpha_1) +$$
$$(0 \quad \alpha_2 \quad 0 \quad -\alpha_2) +$$
$$(0 \quad 0 \quad \alpha_3 \quad -\alpha_3) = (0 \quad 0 \quad 0 \quad 0)$$

$$\Rightarrow \alpha_1 = 0, \ \alpha_2 = 0, \ \alpha_3 = 0.$$

Thus it follows that $\{v_1, v_2, v_3\}$ forms a set of basis vectors of V.

Dimension of $V = 3$.

Gram Schmidt Orthogonalization

Let
$$B = \{v_1, v_2, \ldots, v_m\}$$
be a set of basis vectors of a vector space V. (We would assume $V = \mathbb{R}^n$ for some n).

We want to construct another basis set
$$B_1 = \{u_1, u_2, \ldots, u_m\}$$
of V such that

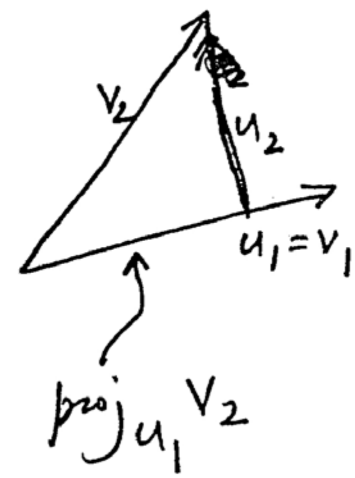
1. $u_i \perp u_j$ for all i, j and $i \neq j$.
2. $[v_1 \ldots v_\ell] = [u_1 \ldots u_\ell]$ for $\ell = 1, 2, 3, \ldots, m$.

One way to obtain B_1 is to choose

$$u_1 = v_1$$

$$u_2 = v_2 - \text{proj}_{u_1} v_2$$

$$= v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1$$



Proceeding similarly we have

$$u_3 = \underbrace{v_3 - \text{proj}_{u_1} v_3 - \text{proj}_{u_2} v_3}$$

Remove from v_3 the component of v_3 along u_1 and u_2.

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2.$$

In general

$$u_\ell = v_\ell - \sum_{j=1}^{\ell-1} \frac{v_\ell \cdot u_j}{u_j \cdot u_j} u_j, \quad \ell = 1, 2, \ldots, m.$$

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Ex 6:

Let us apply the G.S. algorithm on the basis vectors in Ex 5. Recall that

$$V_1 = (1 \quad 0 \quad 0 \quad -1)$$
$$V_2 = (0 \quad 1 \quad 0 \quad -1)$$
$$V_3 = (0 \quad 0 \quad 1 \quad -1)$$

We have

$$u_1 = V_1 = (1 \quad 0 \quad 0 \quad -1)$$
$$u_1 \cdot u_1 = 2$$
$$V_2 \cdot u_1 = 1$$
$$u_2 = (0 \quad 1 \quad 0 \quad -1) - (\tfrac{1}{2} \quad 0 \quad 0 \quad -\tfrac{1}{2})$$
$$= (-\tfrac{1}{2} \quad 1 \quad 0 \quad -\tfrac{1}{2})$$

3.27

$$v_3 \cdot u_1 = 1 \qquad v_3 \cdot u_2 = 1/2$$

$$u_1 \cdot u_1 = 2 \qquad u_2 \cdot u_2 = 3/2$$

$$u_3 = \begin{pmatrix} 0 & 0 & 1 & -1 \end{pmatrix}$$

$$- \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix}$$

$$- \frac{1/2}{3/2} \begin{pmatrix} -\frac{1}{2} & 1 & 0 & -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & -1 \end{pmatrix}$$

$$+ \begin{pmatrix} -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1/6 & -\frac{1}{3} & 0 & 1/6 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} \end{pmatrix}$$

Thus

$$u_1 = \begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix}$$
$$u_2 = \begin{pmatrix} -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \end{pmatrix}$$
$$u_3 = \begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} \end{pmatrix}$$

is a set of orthogonal basis vectors. obtained using the G.S. algorithm.

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Remark:

Many often, we would be interested in an orthonormal basis vectors. This is obtained by normalizing the norm of the vectors u_1, u_2, u_3 to have norm 1. Define

$$w_1 = u_1/\|u_1\| ;\ w_2 = u_2/\|u_2\| ;\ w_3 = u_3/\|u_3$$

The set $\{w_1, w_2, w_3\}$ is an orthonormal basis

Note that

$$\|u_1\| = \sqrt{1+1} = \sqrt{2}.$$

$$\|u_2\| = \sqrt{\frac{1}{4}+1+\frac{1}{4}} = \sqrt{3/2}.$$

$$\|u_3\| = \sqrt{\frac{1}{9}+\frac{1}{9}+1+\frac{1}{9}} = \sqrt{4/3}.$$

$$w_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & -1\end{pmatrix}$$

$$w_2 = \sqrt{\frac{2}{3}}\begin{pmatrix}-\frac{1}{2} & 1 & 0 & -\frac{1}{2}\end{pmatrix}$$

$$w_3 = \sqrt{\frac{3}{4}}\begin{pmatrix}-\frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3}\end{pmatrix}.$$

Orthonormal basis of the vector space